Princess Sumaya University for Technology
Communication Engineering Department
Electromagnetics I
Quiz 2

A charge distribution with spherical symmetry has a density:

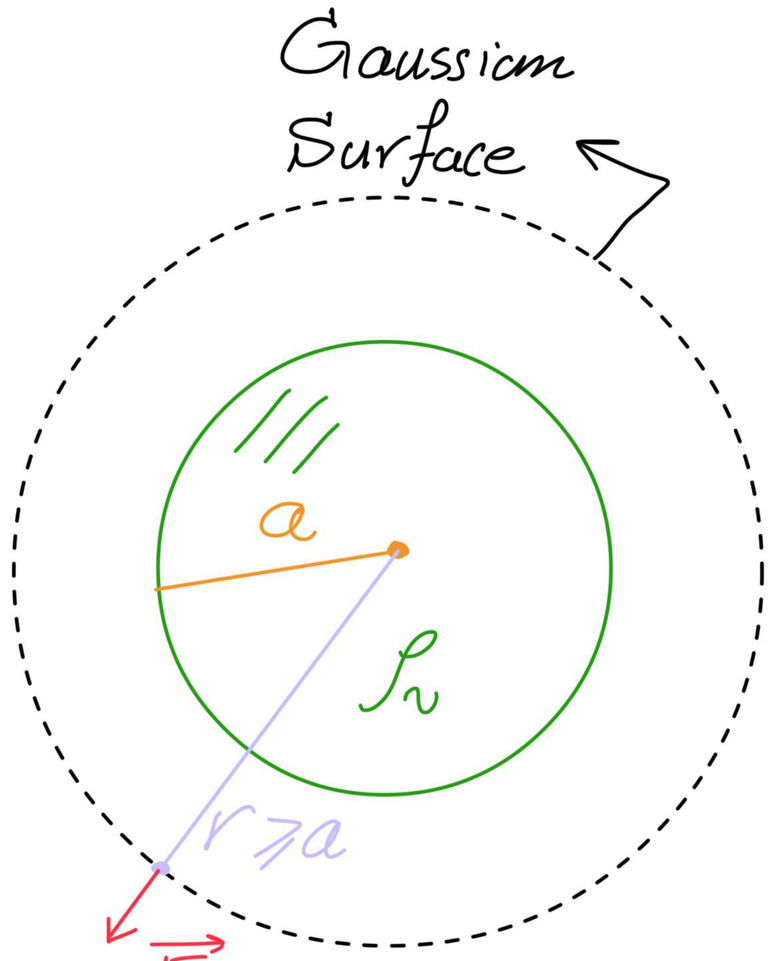
$$\rho_v = \begin{cases} \rho_o(a^2 - r^2), & r < a \\ 0, & r > a \end{cases}$$

a. Find $\boldsymbol{E}$ and V for $r \geq a$.

b. Find $\boldsymbol{E}$ and V for $r \leq a$.

c. Find the total charge.

d. Show that $\boldsymbol{E}$ is maximum when $r = 0.745\ a$.



Soln:

a] Find $\vec{E}$:

Gauss's Law: $Q_{enc} = \int_S \vec{D} \cdot d\vec{S}$

$$Q_{enc} = \int_v \rho_v \, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho_o (a^2 - r^2) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_o \int_0^{2\pi} d\phi \cdot \int_0^{\pi} \sin\theta \, d\theta \cdot \int_0^{a} (a^2 r^2 - r^4) \, dr = 2\pi\rho_o \left[-\cos\theta\right]_0^{\pi} \cdot \left[\frac{a^2 r^3}{3} - \frac{r^5}{5}\right]_0^{a}$$

$$= 2\pi\rho_o (1 - (-1)) \cdot \left(\frac{a^5}{3} - \frac{a^5}{5}\right) = 4\pi\rho_o \cdot \frac{2a^5}{15} = \frac{8\pi\rho_o a^5}{15}$$

$$\int_S \vec{D}\cdot d\vec{S} = D\cdot 4\pi r^2 = 4\pi r^2 \varepsilon_0 E.$$

Now, $Q_{enc} = \int_S \vec{D}\cdot d\vec{S} \Rightarrow \frac{8\pi\rho_0 a^5}{15} = 4\pi r^2 \varepsilon_0 E.$

$$\therefore \vec{E} = \frac{2\rho_0 a^5}{15 r^2 \varepsilon_0}\vec{a}_r \; ; \; r \geq a.$$

Find V:

Method 1:

Since our reference is $r=\infty$: $V = -\int_{\infty}^{r} \vec{E}\cdot d\vec{\ell} = \int_{r}^{\infty} \vec{E}\cdot d\vec{\ell}$

$$= \int_r^{\infty} \frac{2\rho_0 a^5}{15\varepsilon_0 r^2}\vec{a}_r \cdot dr\,\vec{a}_r = \frac{2\rho_0 a^5}{15\varepsilon_0}\int_r^{\infty}\frac{dr}{r^2} = \frac{2\rho_0 a^5}{15\varepsilon_0}\cdot\frac{-1}{r}\Big|_r^{r\to\infty}$$

$$= \frac{2\rho_0 a^5}{15\varepsilon_0 r} \; ; \; r \geq a.$$

Method 2:

$$V = -\int \vec{E}\cdot d\vec{\ell} = -\int \frac{2\rho_0 a^5}{15\varepsilon_0 r^2}\vec{a}_r \cdot dr\,\vec{a}_r = -\frac{2\rho_0 a^5}{15\varepsilon_0}\int \frac{dr}{r^2}$$

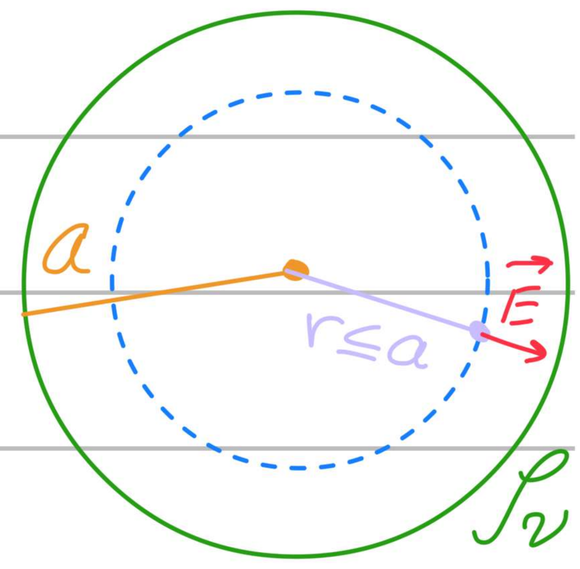
$$= \frac{2\rho_0 a^5}{15\varepsilon_0 r} + A \; ; \; \text{But } A = ?.$$

Since $V(r=\infty) = 0$, (reference), substitute:

$$\lim_{r\to\infty}\frac{2\rho_0 a^5}{15\varepsilon_0 r}+A=0\Rightarrow 0+A=0\Rightarrow A=0.$$

$$\therefore V=\frac{2\rho_0 a^5}{15\varepsilon_0 r}\;;\quad r\geq a.$$

b] Find $\vec{E}$:



$$Q_{enc}=\int_S \vec{D}\cdot d\vec{S}$$

$$Q_{enc}=\int_v \rho_v\,dv=\int_0^{2\pi}\int_0^{\pi}\int_0^{r}\rho_0(a^2-r^2)r^2\sin\theta\,dr\,d\theta\,d\phi.$$

$$=\rho_0\int_0^{2\pi}d\phi\cdot\int_0^{\pi}\sin\theta\,d\theta\cdot\int_0^{r}(a^2r^2-r^4)\,dr$$

$$=\rho_0\cdot 2\pi\cdot 2\cdot\left[\frac{a^2r^3}{3}-\frac{r^5}{5}\right]_0^{r}=4\pi\rho_0\left(\frac{a^2r^3}{3}-\frac{r^5}{5}\right)$$

$$\int_S \vec{D}\cdot d\vec{S}=D\cdot 4\pi r^2=4\pi r^2\varepsilon_0 E.$$

$$Q_{enc}=\int_S \vec{D}\cdot d\vec{S}\Rightarrow \cancel{4\pi}\rho_0\left(\frac{a^2r^3}{3}-\frac{r^5}{5}\right)=\cancel{4\pi r^2}\varepsilon_0 E$$

$$\Rightarrow \rho_0\cancel{r^2}\left(\frac{a^2r}{3}-\frac{r^3}{5}\right)=\cancel{r^2}\varepsilon_0 E\Rightarrow\vec{E}=\frac{\rho_0}{\varepsilon_0}\left(\frac{a^2r}{3}-\frac{r^3}{5}\right)\vec{a_r}\;;\; r\leq a$$

Find V: $-\int\vec{E}\cdot d\vec{l}=-\int\frac{\rho_0}{\varepsilon_0}\left(\frac{a^2r}{3}-\frac{r^3}{5}\right)\vec{a_r}\cdot dr\,\vec{a_r}$

$$=\frac{\rho_0}{\varepsilon_0}\int\left(\frac{r^3}{5}-\frac{a^2r}{3}\right)dr=\frac{\rho_0}{\varepsilon_0}\left[\frac{r^4}{20}-\frac{a^2r^2}{6}\right]+B\;;\;\text{But } B=?$$

Since the voltage is continuous every where, $V(r=a^+) = V(r=\bar{a})$

$$\Rightarrow \frac{2\rho_0 a^5}{15\varepsilon_0 a} = \frac{\rho_0}{\varepsilon_0}\left(\frac{a^4}{20} - \frac{a^4}{6}\right) + B \Rightarrow \frac{2\rho_0 a^4}{15\varepsilon_0} = \frac{\rho_0}{\varepsilon_0}\cdot\frac{-7a^4}{60} + B$$

$$\Rightarrow B = \frac{2\rho_0 a^4}{15\varepsilon_0} + \frac{7\rho_0 a^4}{60\varepsilon_0} = \frac{15\rho_0 a^4}{60\varepsilon_0} = \frac{\rho_0 a^4}{4\varepsilon_0}.$$

$$\therefore V = \frac{\rho_0}{\varepsilon_0}\left(\frac{r^4}{20} - \frac{a^2 r^2}{6}\right) + \frac{\rho_0 a^4}{4\varepsilon_0} = \frac{\rho_0}{\varepsilon_0}\left(\frac{r^4}{20} - \frac{a^2 r^2}{6} + \frac{a^4}{4}\right); \; r \le a.$$

c] $Q_{enc} = \frac{8\pi\rho_0 a^5}{15}$ [Found in Part "a"].

d]

i) $r \ge a$: $\vec{E} = \frac{2\rho_0 a^5}{15\varepsilon_0 r^2}\vec{a_N} \Rightarrow \frac{\partial E}{\partial r} = \frac{-4\rho_0 a^5}{15\varepsilon_0 r^3} = 0.$

$\Rightarrow -4\rho_0 a^5 = 0$?! [Not possible].

ii) $r \le a$:

$$\vec{E} = \frac{\rho_0}{\varepsilon_0}\left(\frac{a^2 r}{3} - \frac{r^3}{5}\right)\vec{a_N} \Rightarrow \frac{\partial E}{\partial r} = \frac{\rho_0}{\varepsilon_0}\left(\frac{a^2}{3} - \frac{3r^2}{5}\right) = 0.$$

$$= \frac{a^2}{3} - \frac{3r^2}{5} = 0 \Rightarrow r^2 = \frac{5a^2}{9} \Rightarrow r = \frac{\sqrt{5}}{3}a.$$

$= r = 0.745a.$ Q.E.D.